Fast practical untangling of simplicial P2 and P3 curvilinear meshes

International Meshing Roundtable 2025

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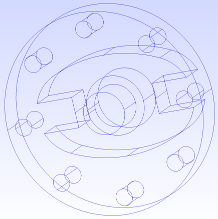
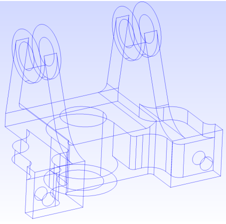
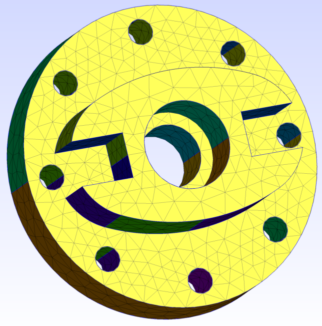
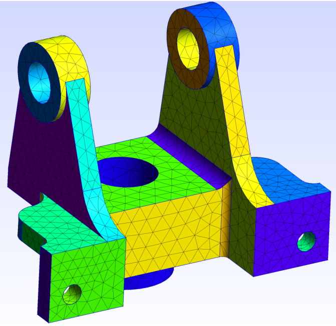
Amaury Johnen – Université catholique de Louvain
Jean-François Remacle – Université catholique de Louvain

GMSH: the meshing software

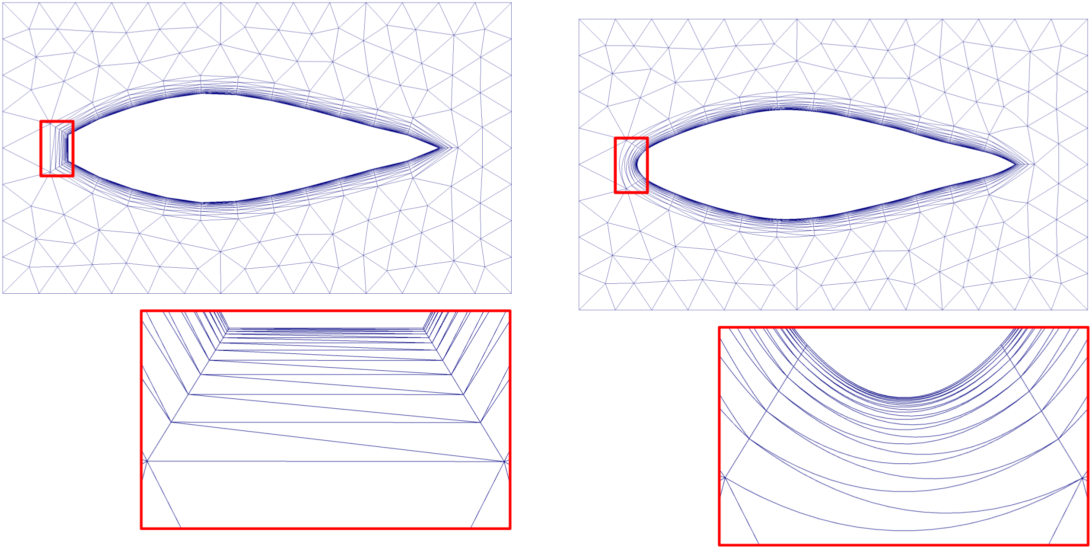








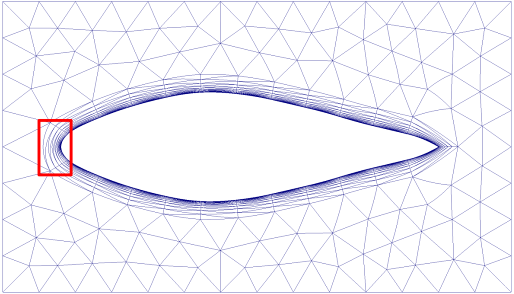
A practical problem: deforming of high order boundary layers without folds

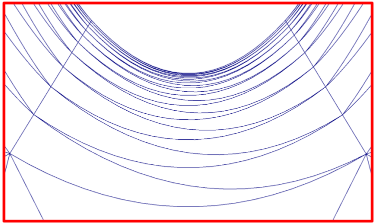


A practical problem: deforming of high order boundary layers without folds

Requirements

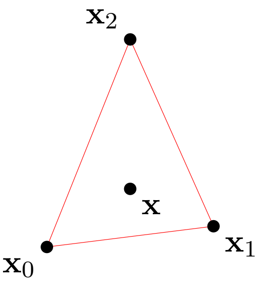
1. Exact boundary
2. Elements should not be inverted
3. Elements should be as close as possible to their target shape





Warm-up:
Untangling and deformation of first order triangles

The P1 triangle



$$\mathbf{x}(\lambda_0, \lambda_1, \lambda_2) = \lambda_0 \mathbf{x}_0 + \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2$$

with $\lambda_0 + \lambda_1 + \lambda_2 = 1, \quad \lambda_i \geqslant 0$

$\mathbf{J} = \det J$ is a constant value over the element (degree 0 polynomial):

$$\mathbf{J}(\lambda_0, \lambda_1, \lambda_2) = \left(\frac{\partial \mathbf{x}}{\partial \lambda_1} - \frac{\partial \mathbf{x}}{\partial \lambda_0} \right) \times \left(\frac{\partial \mathbf{x}}{\partial \lambda_2} - \frac{\partial \mathbf{x}}{\partial \lambda_0} \right)$$

The element is not inverted $\Leftrightarrow \mathbf{J} > 0$

Untangling energy for triangles

Foldover-free maps in 50 lines of code

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FRANÇOIS PROTAIS, NICOLAS RAY, and DMITRY SOKOLOV, Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France

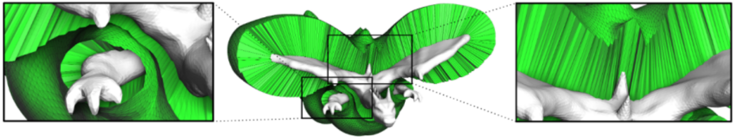


Fig. 1. Our method of constructing injective maps opens a door for a large variety of applications. This figure shows an example of a thick prismatic mesh layer (shown in green) built around a triangulated surface, a very challenging problem for highly curved objects. Thanks to our method, we are able to compute such a layer free of folds and self-intersections.

Mapping a triangulated surface to 2D space (or a tetrahedral mesh to 3D space) is an important problem in geometry processing. In computational physics, untangling plays an important role in mesh generation: it takes a mesh as an input, and moves the vertices to get rid of foldovers. In fact, mesh untangling can be considered as a special case of mapping where the geometry of the object is to be defined in the map space and the geometric domain is not explicit, supposing that each element is regular. In this paper, we propose a mapping method inspired by the untangling problem and compare its performance to the state of the art. The main advantage of

1 INTRODUCTION

Most geometric objects are represented by a triangulated surface or a tetrahedral mesh. The mapping problem consists in generating a 2D or 3D map of these objects. This is a fundamental problem of computer graphics because it is much easier for many applications to work in this map space than to directly manipulate the object itself. To give few examples, texture mapping stores colors of a surface as images in the map space, remeshing uses global maps

Untangling energy for triangles

$$\min_{J} \sum_{t \in T} f(J_t) + \alpha g(J_t)$$

Where:

$$f(J) = \frac{\mathrm{tr}(J^T J)}{\det(J)}$$

$$g(J) = \det J + \frac{1}{\det J}$$

Untangling energy for triangles

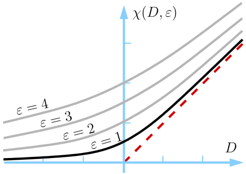
$$\min_J \lim_{\epsilon \to 0} \sum_{t \in T} f_\epsilon(J_t) + \alpha g_\epsilon(J_t)$$

Where:

$$f_\epsilon(J) = \frac{\mathrm{tr}(J^T J)}{\chi(\det(J), \epsilon)}$$

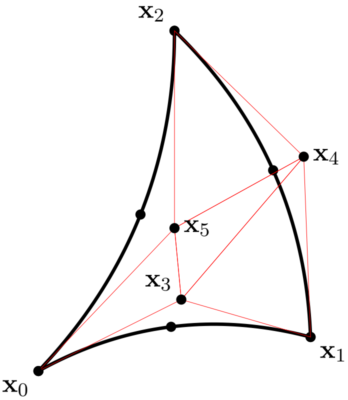
$$g_\epsilon(J) = \frac{\det(J)^2 + 1}{\chi(\det(J), \epsilon)}$$

$$\chi(x, \epsilon) = \frac{x + \sqrt{\epsilon^2 + x^2}}{2}$$



The second order case

The P2 Triangle



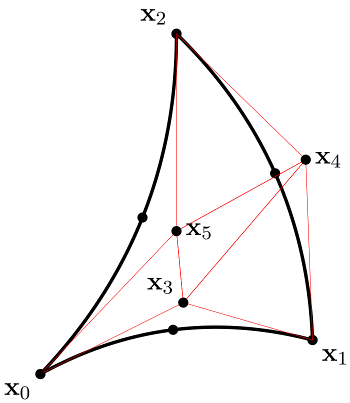
$$\mathbf{x}(\lambda_0, \lambda_1, \lambda_2) = \lambda_0^2\,\mathbf{x}_0 + \lambda_1^2\,\mathbf{x}_1 + \lambda_2^2\,\mathbf{x}_2 + 2\lambda_0\lambda_1\mathbf{x}_3 + 2\lambda_1\lambda_2\mathbf{x}_4 + 2\lambda_2\lambda_0\mathbf{x}_5$$

with $\lambda_0 + \lambda_1 + \lambda_2 = 1, \quad \lambda_i \geqslant 0$

$$\mathbf{J}(\lambda_0, \lambda_1, \lambda_2) = \det J = \left(\frac{\partial \mathbf{x}}{\partial \lambda_1} - \frac{\partial \mathbf{x}}{\partial \lambda_0}\right) \times \left(\frac{\partial \mathbf{x}}{\partial \lambda_2} - \frac{\partial \mathbf{x}}{\partial \lambda_0}\right)$$

is now a polynomial of degree 2 (non-constant!)

The P2 Triangle



$$\begin{aligned}\mathbf{x}(\lambda_0, \lambda_1, \lambda_2) = \lambda_0^2\,\mathbf{x}_0 + \lambda_1^2\,\mathbf{x}_1 + \lambda_2^2\,\mathbf{x}_2 \\ + 2\lambda_0\lambda_1\mathbf{x}_3 + 2\lambda_1\lambda_2\mathbf{x}_4 + 2\lambda_2\lambda_0\mathbf{x}_5\end{aligned}$$

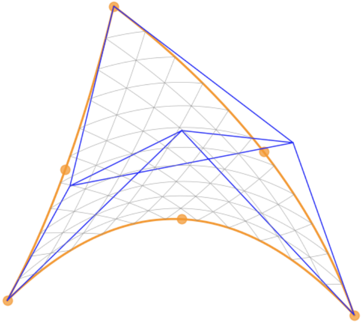
with $\lambda_0 + \lambda_1 + \lambda_2 = 1, \quad \lambda_i \geqslant 0$

$$\mathbf{J}(\lambda_0, \lambda_1, \lambda_2) = \det J = \left(\frac{\partial \mathbf{x}}{\partial \lambda_1} - \frac{\partial \mathbf{x}}{\partial \lambda_0}\right) \times \left(\frac{\partial \mathbf{x}}{\partial \lambda_2} - \frac{\partial \mathbf{x}}{\partial \lambda_0}\right)$$

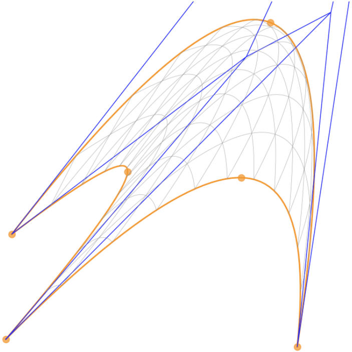
is now a polynomial of degree 2 (non-constant!)

How to enforce $\mathbf{J}(x) > 0$ for all x?

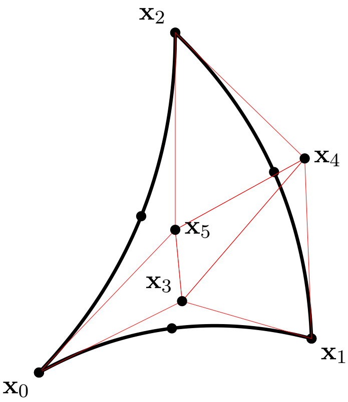
Validity of the Bezier polygon is not necessary nor sufficient



A *valid P2* element where the central triangle is inverted



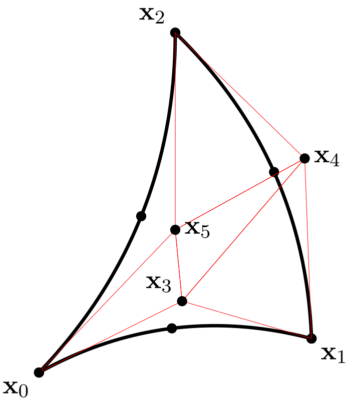
An *invalid* P2 element where all four triangles are valid



$$\begin{aligned}\mathbf{J}(1,0,0) &= \mathbf{J}_0 = 8A_{035} > 0,\\ \mathbf{J}(0,1,0) &= \mathbf{J}_1 = 8A_{143} > 0,\\ \mathbf{J}(0,0,1) &= \mathbf{J}_2 = 8A_{254} > 0.\end{aligned}$$

$$\begin{aligned}\mathbf{J}(\lambda_0,\lambda_1,\lambda_2) = {} & \lambda_0^2\,\mathbf{J}_0 + \lambda_1^2\,\mathbf{J}_1 + \lambda_2^2\,\mathbf{J}_2\\ & + 2\lambda_0\lambda_1\,\mathbf{J}_3 + 2\lambda_1\lambda_2\,\mathbf{J}_4 + 2\lambda_2\lambda_0\,\mathbf{J}_5\end{aligned}$$

where $\mathbf{J}_i =$ "$\mathbf{J}(\mathbf{x}_i)$"



$$\begin{aligned}
\mathbf{J}(1,0,0) &= \mathbf{J}_0 = 8A_{035} > 0,\\
\mathbf{J}(0,1,0) &= \mathbf{J}_1 = 8A_{143} > 0,\\
\mathbf{J}(0,0,1) &= \mathbf{J}_2 = 8A_{254} > 0.
\end{aligned}$$

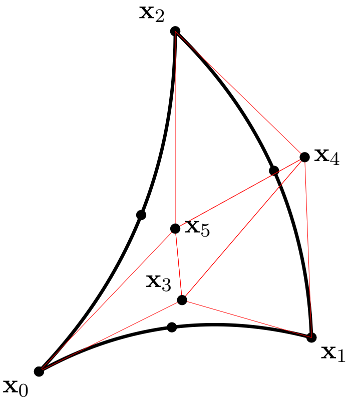
$$\mathbf{J}(\lambda_0,\lambda_1,\lambda_2) = \lambda_0^2\,\mathbf{J}_0 + \lambda_1^2\,\mathbf{J}_1 + \lambda_2^2\,\mathbf{J}_2 + 2\lambda_0\lambda_1\,\mathbf{J}_3 + 2\lambda_1\lambda_2\,\mathbf{J}_4 + 2\lambda_2\lambda_0\,\mathbf{J}_5$$

where $\mathbf{J}_i =$ "$\mathbf{J}(\mathbf{x}_i)$"

Sufficient condition for validity

$\mathbf{J}(x) > 0$ for all x if all the $\mathbf{J}_i$ are positive

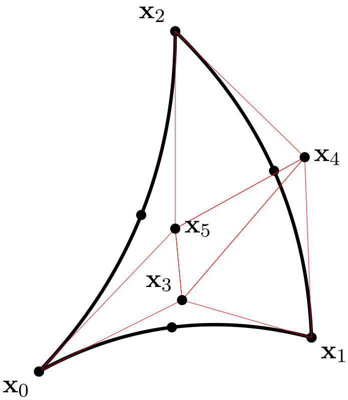
Expansion in the Lagrange basis



$$
\begin{aligned}
\mathbf{J}(1,0,0) &= \mathbf{J}_0 = 8A_{035} > 0, \\
\mathbf{J}(0,1,0) &= \mathbf{J}_1 = 8A_{143} > 0, \\
\mathbf{J}(0,0,1) &= \mathbf{J}_2 = 8A_{254} > 0.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{J}_3 &= 2\mathbf{J}(\tfrac{1}{2},\tfrac{1}{2},0) - \tfrac{1}{2}(\mathbf{J}_0 + \mathbf{J}_1), \\
\mathbf{J}_4 &= 2\mathbf{J}(0,\tfrac{1}{2},\tfrac{1}{2}) - \tfrac{1}{2}(\mathbf{J}_1 + \mathbf{J}_2), \\
\mathbf{J}_5 &= 2\mathbf{J}(\tfrac{1}{2},0,\tfrac{1}{2}) - \tfrac{1}{2}(\mathbf{J}_0 + \mathbf{J}_2),
\end{aligned}
$$

Expansion in the Lagrange basis



Sufficient conditions for $\mathbf{J} > 0$ are therefore:

$$
\begin{aligned}
\mathbf{J}_0 &= A_{035} > 0,\\
\mathbf{J}_1 &= A_{143} > 0,\\
\mathbf{J}_2 &= A_{254} > 0,\\
\mathbf{J}_3 &= A_{034} + A_{153} + A_{013} > 0,\\
\mathbf{J}_4 &= A_{145} + A_{234} + A_{124} > 0,\\
\mathbf{J}_5 &= A_{253} + A_{045} + A_{205} > 0.
\end{aligned}
$$

Optimization: add linear combinations areas to the untangling energy

Minimize:

$$f_\epsilon(J_0) + f_\epsilon(J_1) + f_\epsilon(J_2) \\ + \alpha\left(g_\epsilon(\mathbf{J}_0) + g_\epsilon(\mathbf{J}_1) + g_\epsilon(\mathbf{J}_2)\right) \\ + \alpha\left(g_\epsilon(\mathbf{J}_3) + g_\epsilon(\mathbf{J}_4) + g_\epsilon(\mathbf{J}_5)\right)$$

summed over all elements of the mesh.

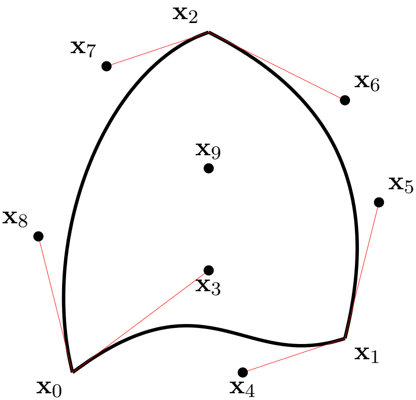
$$\begin{aligned} \mathbf{J}_0 &= A_{035} \\ \mathbf{J}_1 &= A_{143} \\ \mathbf{J}_2 &= A_{254} \\ \mathbf{J}_3 &= A_{034} + A_{153} + A_{013} \\ \mathbf{J}_4 &= A_{145} + A_{234} + A_{124} \\ \mathbf{J}_5 &= A_{253} + A_{045} + A_{205} \end{aligned}$$

$$f_\epsilon(J) = \frac{\text{tr}(J^T J)}{\chi(\mathbf{J}, \epsilon)}$$

$$g_\epsilon(\mathbf{J}) = \frac{\mathbf{J}^2 + 1}{\chi(\mathbf{J}, \epsilon)}$$

Generalizing to higher orders

The P3 Triangle

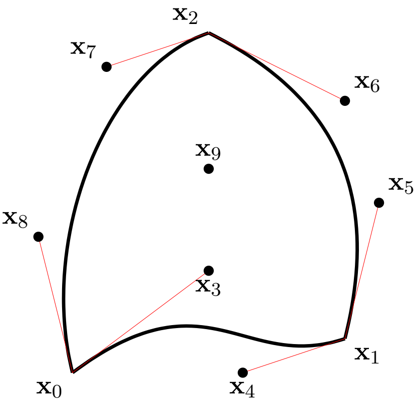


$$\begin{aligned}\mathbf{x}(\lambda_0, \lambda_1, \lambda_2) = {} & \lambda_0^3\mathbf{x}_0 + 3\lambda_0^2\lambda_1\mathbf{x}_3 + 3\lambda_0\lambda_1^2\mathbf{x}_4 + \\ & \lambda_1^3\mathbf{x}_1 + 3\lambda_1^2\lambda_2\mathbf{x}_5 + 3\lambda_1\lambda_2^2\mathbf{x}_6 + \\ & \lambda_2^3\mathbf{x}_2 + 3\lambda_2^2\lambda_0\mathbf{x}_7 + 3\lambda_2\lambda_0^2\mathbf{x}_8 + \\ & 6\lambda_0\lambda_1\lambda_2\mathbf{x}_9.\end{aligned}$$

$$\mathbf{J}(\lambda_0, \lambda_1, \lambda_2) = \left(\frac{\partial \mathbf{x}}{\partial \lambda_1} - \frac{\partial \mathbf{x}}{\partial \lambda_0}\right) \times \left(\frac{\partial \mathbf{x}}{\partial \lambda_2} - \frac{\partial \mathbf{x}}{\partial \lambda_0}\right)$$

is now a polynomial of degree 4

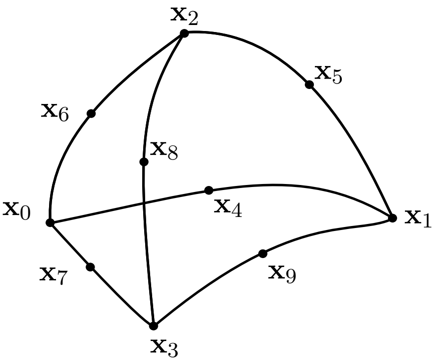
The P3 Triangle



$$
\begin{aligned}
\mathbf{J}_0 &= 18\,A_{038} \\
\mathbf{J}_1 &= 18\,A_{415} \\
\mathbf{J}_2 &= 18\,A_{762} \\
\mathbf{J}_3 &= 9\left(A_{039} + A_{348} + A_{043}\right) \\
\mathbf{J}_5 &= 9\left(A_{419} + A_{345} + A_{143}\right) \\
\mathbf{J}_6 &= 9\left(A_{159} + A_{564} + A_{165}\right) \\
\mathbf{J}_8 &= 9\left(A_{629} + A_{567} + A_{265}\right) \\
\mathbf{J}_9 &= 9\left(A_{279} + A_{786} + A_{287}\right) \\
\mathbf{J}_{11} &= 9\left(A_{809} + A_{783} + A_{087}\right) \\
\mathbf{J}_4 &= 3\left(4\,A_{349} + A_{035} + A_{418} + A_{043} + A_{014}\right) \\
\mathbf{J}_7 &= 3\left(4\,A_{569} + A_{157} + A_{624} + A_{165} + A_{126}\right) \\
\mathbf{J}_{10} &= 3\left(4\,A_{789} + A_{273} + A_{806} + A_{287} + A_{208}\right) \\
\mathbf{J}_{12} &= 3\left(2\,A_{478} - 2\,A_{378} + 2A_{398} + A_{036} + A_{058} - A_{038}\right) \\
\mathbf{J}_{13} &= 3\left(2\,A_{634} - 2\,A_{534} + 2\,A_{594} + A_{158} + A_{174} - A_{154}\right) \\
\mathbf{J}_{14} &= 3\left(2\,A_{856} - 2\,A_{756} + 2\,A_{796} + A_{274} + A_{236} - A_{276}\right)
\end{aligned}
$$

Generalizing to volume meshes

The P2 tetrahedron



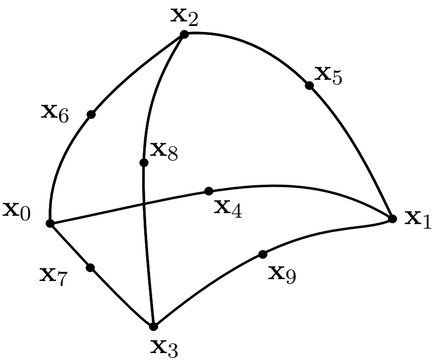
$$\mathbf{x}(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \lambda_0^2\,\mathbf{x}_0 + \lambda_1^2\,\mathbf{x}_1 + \lambda_2^2\,\mathbf{x}_2 + \lambda_3^2\mathbf{x}_3 + 2\lambda_0\lambda_1\mathbf{x}_4 + 2\lambda_1\lambda_2\mathbf{x}_5 + 2\lambda_2\lambda_0\mathbf{x}_6 + 2\lambda_0\lambda_3\mathbf{x}_7 + 2\lambda_2\lambda_3\mathbf{x}_8 + 2\lambda_1\lambda_3\mathbf{x}_9$$

with $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 1, \quad \lambda_i \geqslant 0$

$$\mathbf{J}(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \det\left(\frac{\partial \mathbf{x}}{\partial \lambda_1} - \frac{\partial \mathbf{x}}{\partial \lambda_0}, \frac{\partial \mathbf{x}}{\partial \lambda_2} - \frac{\partial \mathbf{x}}{\partial \lambda_0}, \frac{\partial \mathbf{x}}{\partial \lambda_3} - \frac{\partial \mathbf{x}}{\partial \lambda_0}\right)$$

is a polynomial of degree 3 in $\lambda_0, \lambda_1, \lambda_2, \lambda_3$

The P2 tetrahedron



$$\begin{aligned}
\mathbf{J}_0 &= 48\, V_{0467} \\
\mathbf{J}_4 &= 16\left(V_{0469} + V_{0457} + V_{0167} - V_{0467}\right) \\
\mathbf{J}_{16} &= 8\left(V_{0458} - V_{0456} + V_{0429} - V_{0469}\right. \\
&\quad +V_{0467} - V_{0468} + V_{0127} - V_{0427} \\
&\quad \left.+V_{0168} - V_{0167} + V_{0569} + V_{4567}\right)
\end{aligned}$$

and 17 other coefficients... (20 total)

The general case

1. Get an expression of the Jacobian $\mathbf{J}$ (polynomial of degree $d(n-1)$)
2. Express the coefficients of $\mathbf{J}$ in the Lagrange basis as sums of triangle areas
3. Minimize the untangling energy (take P1 element as the reference element)

The general case

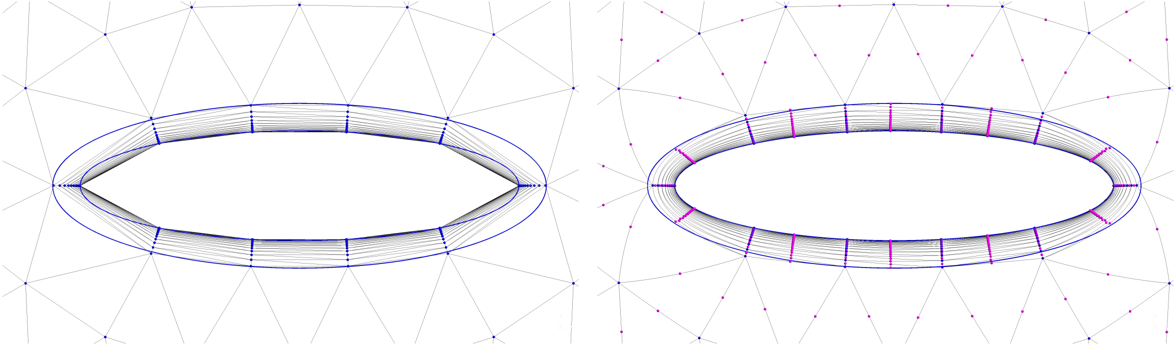
1. Get an expression of the Jacobian $\mathbf{J}$ (polynomial of degree $d(n-1)$)
2. Express the coefficients of $\mathbf{J}$ in the Lagrange basis as sums of triangle areas
3. Minimize the untangling energy (take P1 element as the reference element)

The number of areas to evaluate for order n grows as $\mathcal{O}(n^4)$

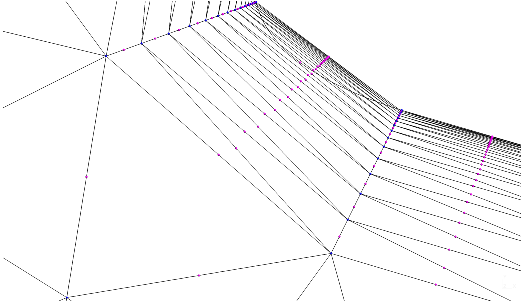
P2 triangle	P3 triangle	P4 triangle	
6 equations	15 equations	28 equations	etc.
12 simplexes	54 simplexes	154 simplexes	

Results

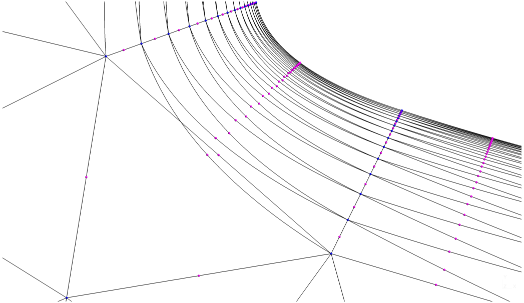
Results: Ellipse



Initialization trick for boundary layers

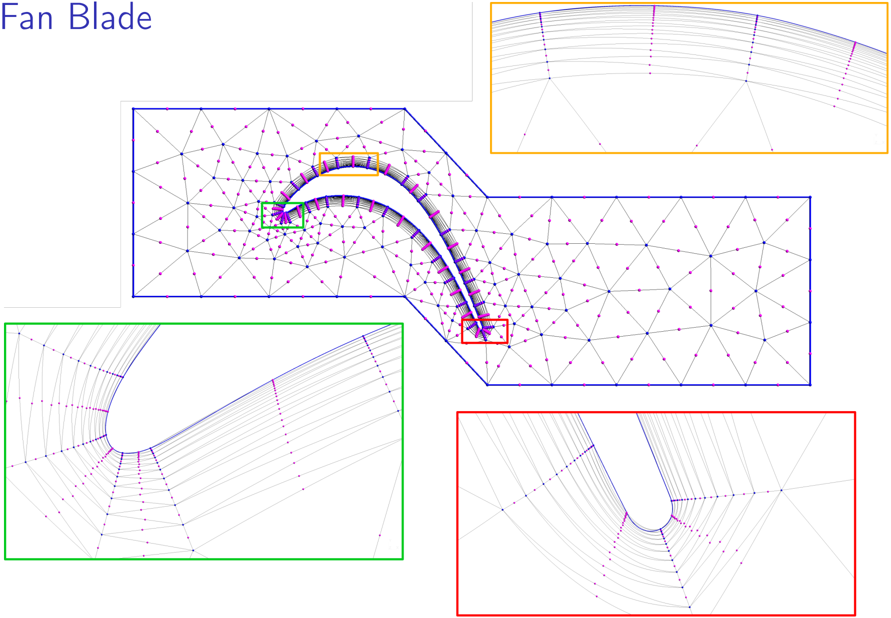


Flipped boundary element initialization (slow)

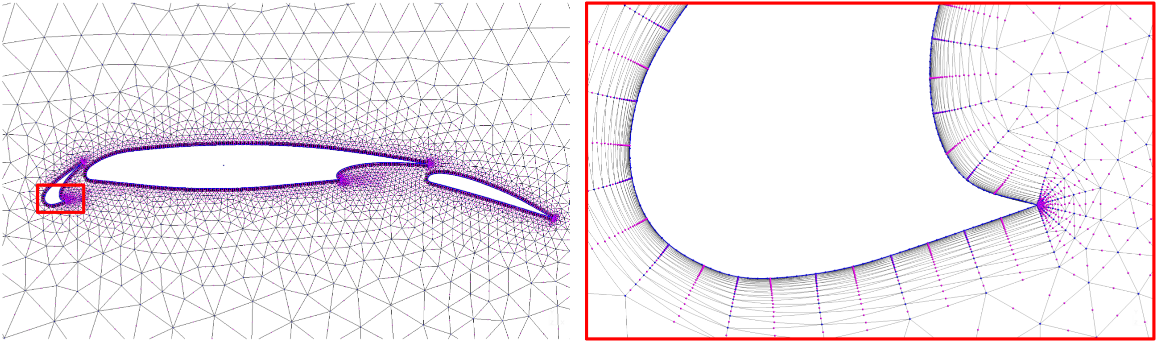


Shifted boundary layer initialization (faster)

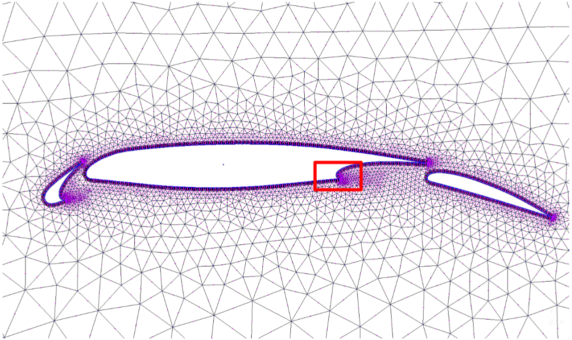
Results: Fan Blade

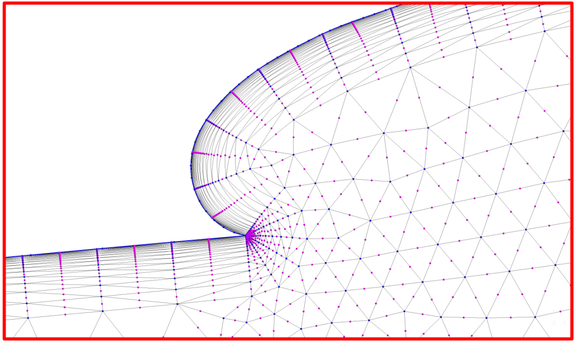


Results: Three Components Wing

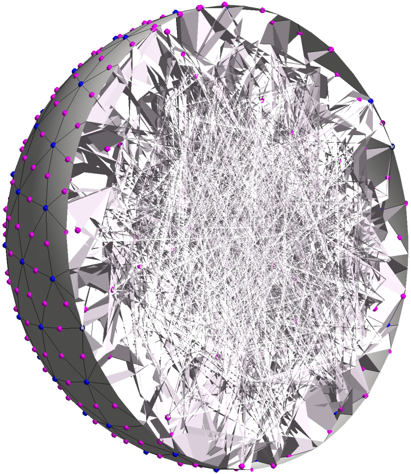


Results: Three Components Wing

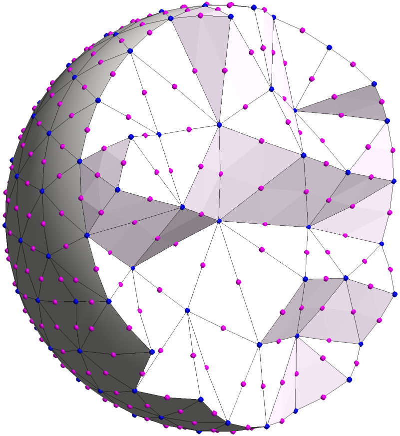




Stress test for tetrahedral mesh



Tangled initial sphere
(random point positions)



Untangled sphere

Questions?